

Amplitude death in coupled chaotic oscillators

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Amplitude death can occur in chaotic dynamical systems with time-delay coupling, similar to the case of coupled limit cycles. The coupling leads to stabilization of fixed points of the subsystems. This phenomenon is quite general, and occurs for identical as well as nonidentical coupled chaotic systems. Using the Lorenz and Rössler chaotic oscillators to construct representative systems, various possible transitions from chaotic dynamics to fixed points are discussed.

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I. INTRODUCTION

Amplitude death occurs when coupled oscillators drive each other to fixed points and stop oscillating [1–3]. Initial studies of this phenomenon were on systems wherein the interaction between the subsystems was assumed instantaneous. In the absence of coupling the dynamics in the individual subsystems were on limit cycles of different periods, near Hopf bifurcations. New stable fixed points are created by the coupling, and these become the attractors of the dynamics.

There has been considerable recent interest [4] on the effects of coupling in a variety of nonlinear systems, not only in the physical sciences, but also in biological and social systems in which in addition to the possibility of chaotic dynamics, complex phenomena such as synchronization, spatiotemporal intermittency, hysteresis, etc. have been investigated. Theoretical as well as experimental studies of coupled systems have addressed a number of issues which include phase-shifting, phase locking [2], as well as amplitude death.

It is often necessary to take account of inherent time delays in the coupling, and Reddy *et al.* [5] investigated the collective dynamical behavior of limit-cycle oscillators interacting diffusively through time delay at a Hopf bifurcation (see also Sec. III). They observed amplitude death of oscillations, regardless of the frequencies of individual oscillators, both theoretically and experimentally. In other work, amplitude death phenomenon has been studied theoretically in distributed time delay [6] and in a ring of delayed-coupled [7] oscillators and experimentally in coupled electronic circuit [8] and coupled thermo-optical oscillators [9]. Phenomena similar to amplitude death are also observed in time-delayed self-feedback in limit-cycle oscillators [10] and chaotic oscillators [11].

That this phenomenon can occur for coupled *chaotic* oscillators as well is the main result of the present work. By introducing time delay in the coupling of chaotic subsystems, irrespective of their type, one can achieve amplitude death. This transition can be direct, namely from chaotic motion to fixed points, or indirect, when first a pair of limit cycles (rather than a pair of fixed points) are stabilized through the

coupling. The motion then asymptotically becomes periodic, and the subsystems oscillate at a common frequency. As a function of the time delay, however, this limit cycle motion can go from being in-phase to being out-of-phase. This transition is signaled by a dramatic change in two indicators: the phase difference between oscillators and their common frequency. Further variation of the time delay eventually leads to amplitude death via stabilization of new fixed points.

In the following section we show that identical coupled chaotic systems with matching or mismatched parameters, and indeed even completely different chaotic systems, all show amplitude death. The in-phase to out-of-phase transition in coupled limit cycle oscillators can be treated analytically. We estimate the common frequency for a particular case in Sec. III, and although similar analysis cannot be carried out for chaotic oscillators, this provides some insight into how the phenomenon arises more generally. The paper concludes with a discussion and summary in Sec. IV.

II. AMPLITUDE DEATH

The basic phenomenology of amplitude death in limit-cycle systems has been described in some detail [2,5]. We consider the general case of chaotic coupled systems,

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}_{\mathbf{x}}(\mathbf{x}) + \mathbf{g}_{\mathbf{x}}(\mathbf{y}(t - \tau), \mathbf{x}(t)), \\ \dot{\mathbf{y}} &= \mathbf{f}_{\mathbf{y}}(\mathbf{y}) + \mathbf{h}_{\mathbf{y}}(\mathbf{x}(t - \tau), \mathbf{y}(t)),\end{aligned}\tag{1}$$

where \mathbf{x} and \mathbf{y} denote the variables of the two subsystems. The dynamical equations are specified by $\mathbf{f}_{\mathbf{x}}$ and $\mathbf{f}_{\mathbf{y}}$, respectively, and $\mathbf{g}_{\mathbf{x}}$ and $\mathbf{h}_{\mathbf{y}}$ specify the couplings, τ being the time delay.

Several scenarios are possible. In the simplest case, the subsystems specified by the variables \mathbf{x} and \mathbf{y} can be identical. This is dealt with in Sec. II A below. By altering the parameters, the two subsystems can be made nonidentical, and this case is treated in Sec. II B. Finally, the two chaotic systems can be completely different, and this case is presented in Sec. II C. In all cases we demonstrate that amplitude death is possible, though the mechanism in each case is somewhat different. For simplicity, the results are presented through applications to specific Rössler or Lorenz dynamical systems which have been extensively studied in the context

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of chaotic dynamics [12]. The present results are mainly numerical since analytic results for such nonlinear systems are difficult to obtain.

In each of the cases, we address the possibility of amplitude suppression, namely the stabilization of limit cycles. For this phenomenon, some analysis is possible, and is presented in Sec. III.

A. Identical chaotic subsystems

The Rössler system [13] is a simple mathematical model of chemical kinetics that incorporates reaction-diffusion, and has been extensively studied in the past two decades as one of the simplest chaotic flows. Consider two identical Rössler oscillators denoted by variables \mathbf{x} and \mathbf{y} ,

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -x_2 - x_3, \\ \frac{dx_2(t)}{dt} &= x_1 + ax_2 + \epsilon[y_2(t-\tau) - x_2(t)], \\ \frac{dx_3(t)}{dt} &= b + x_3(x_1 - c_1), \\ \frac{dy_1(t)}{dt} &= -y_2 - y_3, \\ \frac{dy_2(t)}{dt} &= y_1 + ay_2 + \epsilon[x_2(t-\tau) - y_2(t)], \\ \frac{dy_3(t)}{dt} &= b + y_3(y_1 - c_2), \end{aligned} \quad (2)$$

which are symmetrically coupled through x_2 and y_2 . We take numerical values of the parameters as $a=b=0.1$ and $c_1=c_2=14$ with the coupling parameters ϵ and the delay τ being treated as variable.

In absence of coupling, $\epsilon=0$, the subsystems, which are chaotic [13], evolve independently. For finite coupling strength various states of motion are observed. These are indicated in the schematic Fig. 1(a) within a representative range of coupling parameters ϵ and τ . The region marked C which is shaded black, corresponds to chaotic states while the white region shows the regions of regular behavior, namely periodic (P), fixed point (FP), and hypertorus (HT) [14] dynamics. Numerical details are given in [15,16]. Finer analysis at a fixed value of coupling strength, $\epsilon=0.5$, is shown in Fig. 1(b). The three largest Lyapunov exponents are shown as a function of the time delay τ . The dotted line in Fig. 1(a) indicates the transition where the largest Lyapunov exponent (λ_1) becomes negative [see Fig. 1(b)]. The vertical arrow in Fig. 1(b) denotes the parameter at which the slope of λ_1 changes sign. In Fig. 1(a), the locus of this point is plotted as a dashed line. The solid line in Fig. 1(a) shows the transition from periodic solution to the hypertorus where $\lambda_1 \approx \lambda_2 \approx 0$.

In the region marked P in Fig. 1(b), the dynamics of the two subsystems settles onto limit cycles. Trajectories [17] of

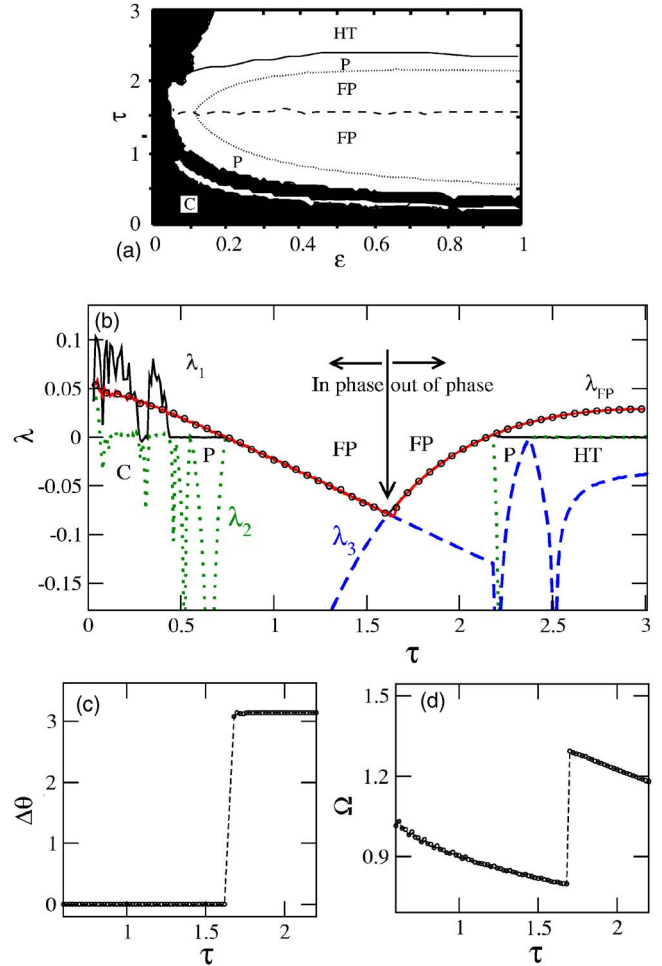


FIG. 1. (Color online) (a) Schematic phase diagram of identical Rössler oscillators, Eq. (2), in the ϵ - τ plane for $c_1=c_2=18$. In the dark region the motion is chaotic (C) while in the white region there are regular states of different kind, e.g., fixed point (FP), periodic cycles (P), and hypertorus (HT). Numerical details are given in [15]. The dotted line indicates the locus where the largest Lyapunov exponent (λ_1) becomes negative. (b) The spectrum of Lyapunov exponents (black, green, and blue correspond to λ_1 , λ_2 , and λ_3 , respectively) as a function of time delay τ at fixed coupling strength $\epsilon=0.5$. The vertical arrow in (a) shows the parameter value, τ_c , when the motion goes from being in-phase to out-of-phase. The largest Lyapunov exponent of the fixed point is shown in red. In the amplitude death range all the Lyapunov exponents are negative and $\lambda_1=\lambda_2=\lambda_{FP}$. The transition is from limit cycle to a fixed point. (c) The phase difference between oscillators with time delay τ , which before and after the transition are equal to 0 and π . (d) Numerically calculated common frequency, Ω , as a function of the time delay τ .

the individual oscillators in the x_1 - x_2 plane with $\tau=0.6$ are shown in Figs. 2(a) and 2(b), respectively. Here the trajectories of both the oscillators overlap since they are in complete synchronization. Further, these are also in-phase since the phase difference is zero. There is also another periodic region between the FP and HT motions [between dotted and solid lines in Fig. 1(a)] where trajectories of individual oscillators at $\tau=2.25$ are shown in Figs. 2(c) and 2(d). In the latter case both the oscillators show the same periodic motion (frequency locked states) [see Fig. 2(c)] but they are

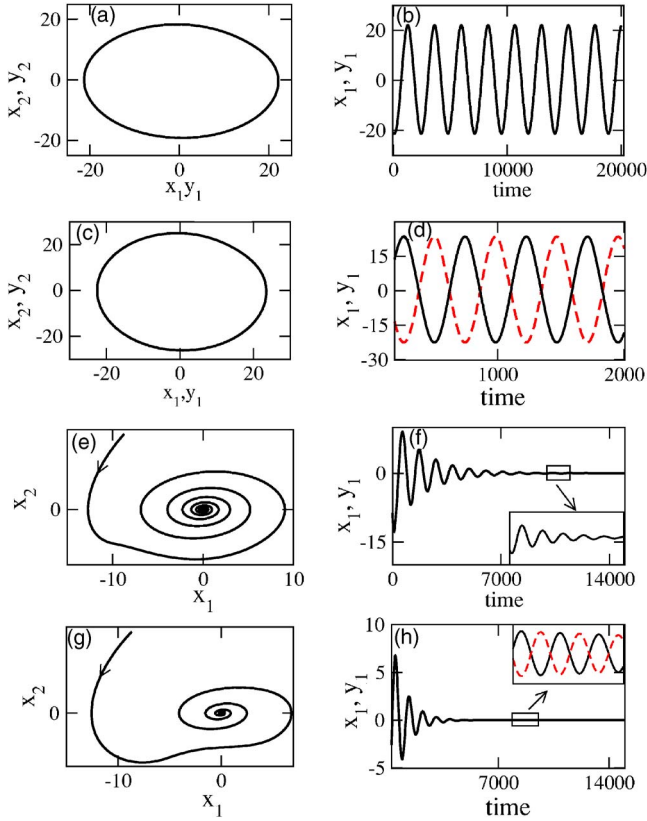


FIG. 2. (Color online) Trajectories of oscillator 1 (solid line) and 2 (dashed line) in phase space (left panel) and with time (right panel) (a) and (b) at $\tau=0.6$, (c) and (d) at $\tau=2.25$, (e) and (f) at $\tau=1.5$, and (g) and (h) at $\tau=2$, respectively.

out-of-phase [Fig. 2(d)], with a constant phase difference of π .

Both types of periodic motions (either in- or out-of-phase) can arise from the following reasons [18]. Either the coupling stabilizes one of the (infinitely many) unstable periodic orbits (UPO) [19] which are embedded in the chaotic set, or creates new periodic solutions with time period (T) different from the time delay, i.e., $\tau \neq T$ (the perturbation term does not vanish here). In the former case only those UPOs will be stabilized whose time periods are equal to the time delay, namely $\tau=T$. A trajectory will settle on an UPO and remain there since the feedback function becomes zero: the coupling term vanishes when $y_2(t-\tau)$ becomes equal to $x_2(t)$. This implies that the perturbation does not change the original solution. In the latter situation, the effect of time delay is shown in Figs. 2(a)–2(d).

As τ is increased further [between the dotted lines in Fig. 1(a), where the largest Lyapunov exponent, λ_1 , is negative as can be seen in Fig. 1(b)] fixed point (FP) solutions are obtained. Examples of such solutions, before and after the marked arrow in Fig. 1(b) are shown in Figs. 2(e)–2(f) and Figs. 2(g)–2(h), respectively, for $\tau=1.5$ and 2. In both cases transient trajectories start spiraling into one of the fixed points $x_{1*}=y_{1*}=ax_{3*}$, $x_{2*}=y_{2*}=-x_{3*}$, $x_{3*}=y_{3*}$, $x_{3*}=[c-\sqrt{(c^2-4ab)}/2a]$. In these regions, since the subsystems are identical, the coupling term vanishes and the fixed points remain the same as that of the uncoupled systems. The largest

Lyapunov exponent for the fixed point solution (λ_{FP}), the position of which does not change with time delay at a given value of the coupling strength, is also shown in Fig. 1(b) (red line with circles). Since this is negative in the FP region (where $\lambda_1=\lambda_2=\lambda_{FP}$), the fixed point is stable. Any attempt by the subsystems to move from this state increases the coupling term and brings the entire system back to the fixed point. Thus by using time-delay coupling, amplitude death phenomena can be observed even in chaotic systems in a manner which is very similar to that in limit cycles [2,5].

In the transition from chaos to amplitude death that occurs via a limit cycle, the mechanism is the same as that of the coupled limit cycle case [2,5] (see also Sec. III), i.e., the transition from limit cycle to FP takes place because a pair of complex eigenvalues cross the imaginary axis from right to left and the Lyapunov exponent λ_{FP} becomes negative. Simultaneously the stability of the fixed point is lost as a pair of eigenvalues cross the axis from left to right. Thus in this case amplitude death is initiated at a Hopf bifurcation.

There are, however, some differences in the manner in which the fixed points are reached. The largest Lyapunov exponent has a change in slope at the point τ_c marked by the arrow, and differences can be seen in the phase relationship of the two oscillators: for $\tau < \tau_c$ [inset of Fig. 2(f)] the two are in-phase, while for $\tau > \tau_c$ [inset of Fig. 2(h)], they are out-of-phase. Shown in Fig. 1(c) is the phase difference ($\Delta\theta$) between oscillators which is defined as $\Delta\theta = \langle |\theta_1(t) - \theta_2(t)| \rangle$ where $\langle \cdot \rangle$ denotes the average over time while $\theta_1(t) \sim \tan^{-1}[x_2(t)/x_1(t)]$ and $\theta_2(t) \sim \tan^{-1}[y_2(t)/y_1(t)]$ [20,21]. This clearly indicates that below τ_c the phase difference is zero while after it is π . We also observe [see Figs. 2(b), 2(d), 2(f), and 2(h)] that the two oscillators are frequency-locked. Their common frequency, Ω (measured from the peak-to-peak separation [20]) is shown in Fig. 1(d) where a steplike change is observed at τ_c . This behavior is analyzed in a simpler system in the following section.

The change in the phase difference and common frequency occurs at parameter values corresponding to maximal stability, namely when the Lyapunov exponent is at a local minimum. While details of the precise mechanism in chaotic systems needs further analysis, it can be easily verified that this behavior is quite general, and a similar transition occurs in a variety of model systems.

Consider coupled identical chaotic Lorenz oscillators [23],

$$\frac{dx_1(t)}{dt} = -\sigma(x_1 - x_2),$$

$$\frac{dx_2(t)}{dt} = -x_1x_3 - x_2 + r_1x_1 + \epsilon[y_2(t-\tau) - x_2(t)],$$

$$\frac{dx_3(t)}{dt} = x_1x_2 - dx_3,$$

$$\frac{dy_1(t)}{dt} = -\sigma(y_1 - y_2),$$

(3)

$$\frac{dy_2(t)}{dt} = -y_1 y_3 - y_2 + r_2 y_1 + \epsilon [x_2(t - \tau) - y_2(t)],$$

$$\frac{dy_3(t)}{dt} = y_1 y_2 - d y_3,$$

with $\sigma = 10, r_1 = r_2 = 28, d = 8/3$. The phase diagram as a function of the coupling parameters τ and ϵ is shown in Fig. 3(a), while the Lyapunov exponents with time delay τ at fixed $\epsilon = 0.5$ are shown in Fig. 3(b). The step transition in phase difference and common frequency are shown in Figs. 3(c) and 3(d), respectively. The behavior is very similar to the Rössler case, Fig. 1. Transient trajectories that go to the fixed point across the marked vertical arrow at $\tau = 0.12$ and 0.17 are shown in Figs. 3(e) (in-phase) and 3(f) (out-of-phase), respectively. The fixed point at $[\pm\sqrt{d(r-1)}, \pm\sqrt{d(r-1)}, r-1]$ is stable in the region marked FP, and has a negative Lyapunov exponent, λ_{FP} [24]. In this region $\lambda_1 = \lambda_2 = \lambda_{FP}$. Thus in this regime amplitude death also occurs.

However, there is a difference in manner of transition from that of Fig. 1. In the case of coupled limit cycle [5] or Rössler system, Eq. (2), transition is from periodic to fixed point while in this case it happens directly from chaotic dynamics to a fixed point. This implies that although neither UPOs are stabilized, nor new periodic solutions are created via the time-delay interaction, the fixed point is stabilized directly and this leads to an abrupt change in the largest Lyapunov exponent of the system, λ_1 . Such a transition, which has not been discussed earlier, suggests that a new mechanism may be operative in amplitude death.

B. Nonidentical chaotic subsystems

We consider the effect of a difference in parameters between the two subsystems. This is particularly important with respect to experimental realization of such phenomena since in practice it is impossible to ensure that all parameters of two different systems are exactly equal. Introduce a mismatch in the parameters of the Rössler system, Eq. (2). For $c_1 = 14$ and $c_2 = 18$, various possible states are shown in Fig. 4(a). Other parameters are the same as in Fig. 1. A major difference, compared with Fig. 1 is the absence of hyperbolic dynamics. The spectrum of Lyapunov exponents at $\epsilon = 0.5$ are shown in Fig. 4(b); in the fixed point region all the Lyapunov exponents are negative. The periodic states at $\tau = 0.25$ and $\tau = 2.5$ are shown in Figs. 4(e) and 4(f) where the motions are in- and out-of-phase, respectively (these are in the frequency locked regime where time periods are the same for both the oscillators).

Shown in Figs. 4(g) and 4(h) are transient trajectories at $\tau = 1$ and $\tau = 2$, respectively. The insets are enlarged views of the corresponding transient and fixed point motions. Here coupling introduces a *new* set of fixed points and the system settles in one of these. The fixed points for Eq. (2) simply turn out to be

$$x_{1\star} = a x_{3\star} - \epsilon (x_{3\star} - y_{3\star}),$$

$$x_{2\star} = -x_{3\star},$$

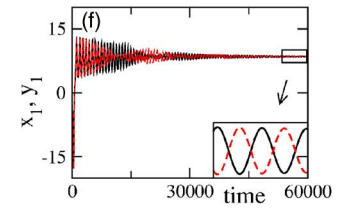
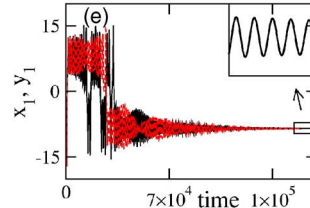
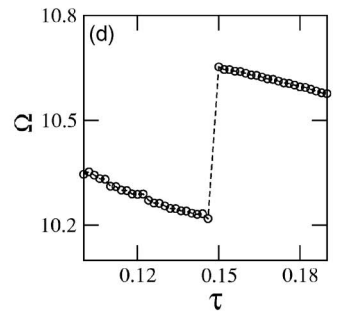
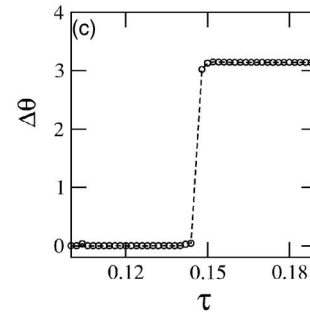
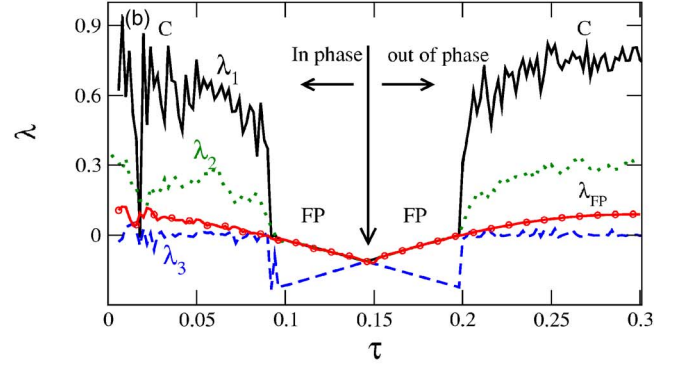
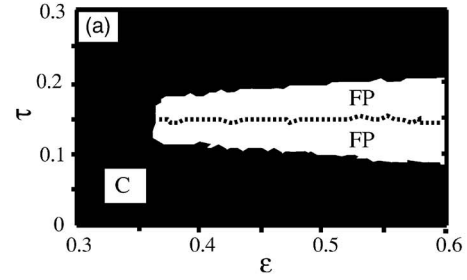


FIG. 3. (Color online) (a) Schematic phase diagram for the Lorenz system Eq. (3) in the ϵ - τ plane. (b) Spectrum of Lyapunov exponents (black, green, and blue corresponding to λ_1, λ_2 , and λ_3 , respectively) as a function of the time delay, τ , at fixed coupling strength $\epsilon = 0.5$. The other details remains the same as that in Fig. 1. In the amplitude death range all the Lyapunov exponents are negative and $\lambda_1 = \lambda_2 = \lambda_{FP}$; the transition is directly from chaos to a fixed point. (c) The phase difference between oscillators with time delay τ . The phase difference before and after the transition are equal to 0 and π . (d) Numerically calculated common frequency, Ω . Trajectories of oscillators 1 (solid line) and 2 (dashed line) with time at (e) $\tau = 0.12$ and (f) $\tau = 0.17$. The inset figures are the enlarged view of the corresponding marked boxes.

$$x_{3\star} = [(\epsilon - a)y_{3\star}^2 + c_2 y_{3\star} - b] / \epsilon y_{3\star},$$

$$y_{1\star} = a y_{3\star} - \epsilon (y_{3\star} - x_{3\star}), \quad (4)$$

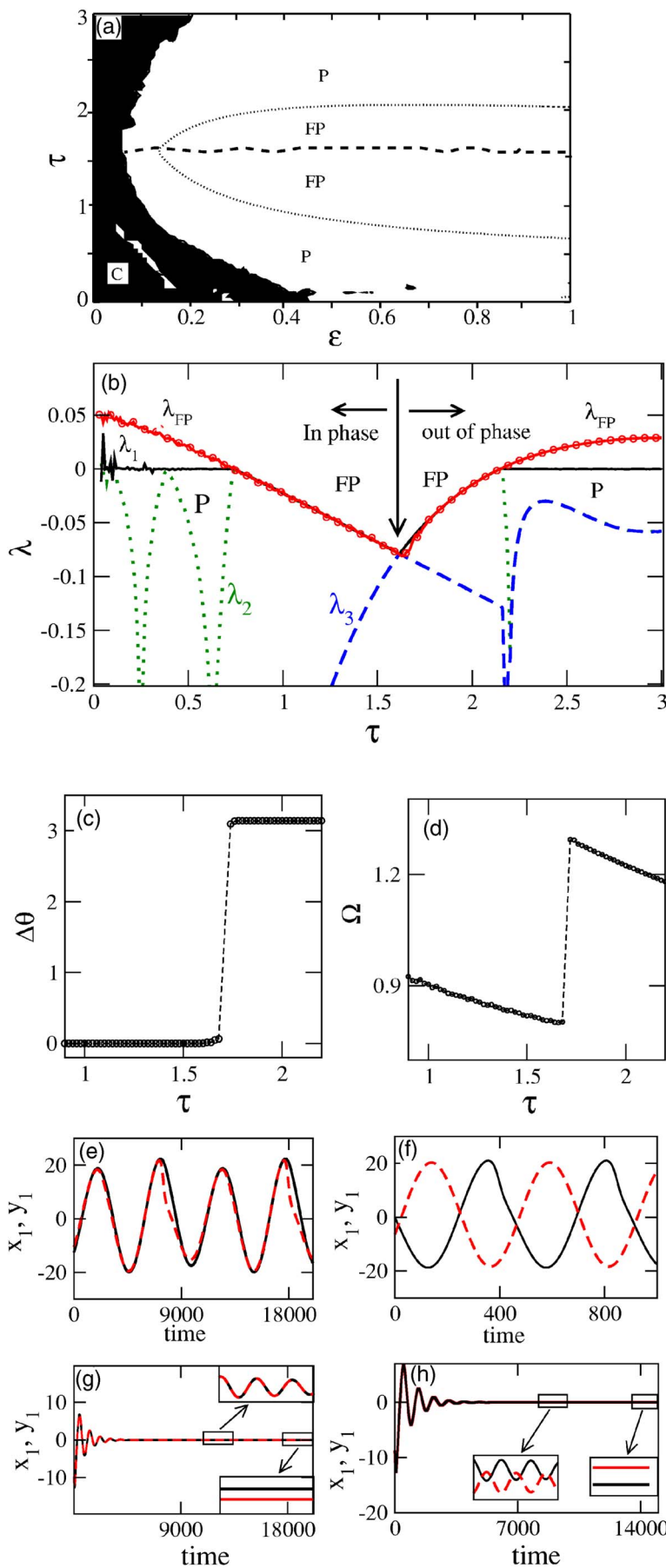


FIG. 4. (Color online) (a) Schematic phase diagram of Eq. (2) in the ϵ - τ plane. Parameters values are $c_1=14$ and $c_2=18$. (b) Spectrum of Lyapunov exponent (black, green, and blue correspond to λ_1, λ_2 , and λ_3 , respectively) as a function of the time delay, τ , at fixed coupling strength $\epsilon=0.5$. Other details remain the same as that of Fig. 1. In amplitude death range all Lyapunov exponents are negative and $\lambda_1=\lambda_2=\lambda_{FP}$. This is also a chaos to FP transition as in Fig. 1(c). The phase difference between oscillators with time delay τ . The phase difference before and after the transition are near to 0 and π , respectively. (d) Numerically calculated common frequency, Ω . Trajectories of oscillator 1 (solid line) and 2 (dashed line) for the nonidentical case with time (e) at $\tau=0.25$, (f) at $\tau=2.5$, (g) at $\tau=1$, and (h) at $\tau=2$. The inset figures are the enlarged views of the corresponding marked boxes.

$$y_{2\star} = -y_{3\star},$$

where $y_{3\star}$ is the root of the polynomial

$$b + y_{3\star}[ay_{3\star} - \epsilon(y_{3\star} - x_{3\star}) - c_2] = 0.$$

The largest Lyapunov exponent λ_{FP} (red line with circles), of the fixed point $x_{1\star}=0.00135\dots$, $x_{2\star}=-0.0056\dots$, $x_{3\star}=0.0056\dots$, $y_{1\star}=-7.914\dots \times 10^{-5}$, $y_{2\star}=-0.00714\dots$, and $y_{3\star}=0.00714\dots$ is shown in Fig. 4(b). Here the transition from chaos to a fixed point occurs indirectly, namely via a limit cycle at a Hopf bifurcation as in the case of identical coupled Rössler oscillators, Fig. 1.

Similar step transitions in the phase difference and common frequency as in Figs. 1(c), 1(d), 3(c), and 3(d) are also observed here in Figs. 4(c) and 4(d). However, the phase differences are only ≈ 0 and $\approx \pi$ before and after the transition (marked by the arrow).

Amplitude death phenomena thus also occurs in nonidentical chaotic subsystems, though there can be subtle differences in the nature of the final state which can have either in-phase [2,5] or out-of-phase [26] similar to the identical subsystem case. Similar results have been obtained [25] for the case of mismatched coupled Lorenz systems.

C. Mixed chaotic subsystems

Since amplitude death occurs for interacting oscillators with very different time periods, it is natural to consider the situation when the two subsystems are dynamically distinct. This is of even greater relevance experimentally, since a variety of different oscillator systems can be mutually coupled in a given natural situation.

To examine the behavior obtained when distinct oscillators are coupled, I study coupled Rössler and Lorenz oscillators,

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -x_2 - x_3, \\ \frac{dx_2(t)}{dt} &= x_1 + ax_2 + \epsilon[y_2(t-\tau) - x_2(t)], \\ \frac{dx_3(t)}{dt} &= b + x_3(x_1 - c_1), \\ \frac{dy_1(t)}{dt} &= -\sigma(y_1 - y_2), \\ \frac{dy_2(t)}{dt} &= -y_1y_3 - y_2 + r_2y_1 + \epsilon[x_2(t-\tau) - y_2(t)], \\ \frac{dy_3(t)}{dt} &= y_1y_2 - dy_3, \end{aligned} \quad (5)$$

where parameters $c_1=18$ and $r_2=28$. Other parameters are the same as in Eqs. (2) and (3). A typical phase diagram is shown in Fig. 5(a), and the Lyapunov exponent as a function of delay τ is shown in Fig. 5(b). Various possible states are

indicated: the region of fixed point motion, i.e., amplitude death phenomena, is observed in the range $\tau \sim 0.3-0.52$ (where $\lambda_1 = \lambda_{FP}$). Trajectories for $\tau=0.35$ and 0.45 in amplitude death region are shown in Figs. 5(e) and 5(f), respectively. The inset figures indicate that both the subsystems are neither completely in nor completely out of phase, i.e., phase differences are away from zero or near π [see Figs. 5(c) and 5(d)] due to the completely distinct natures of the chaotic dynamics of the individual subsystems. Even though the subsystems are distinct, it has been possible to observe indications of a vestige of the transition in the phase difference when Lyapunov exponent is at a local minimum [25].

Regions, denoted by R , where wild fluctuations [see Fig. 5(a)] in the Lyapunov exponent occur are indicative of the presence of coexisting attractors with complicated basins [25,27]. I have verified the presence of riddled [28] basins [results are not presented here] in this region, and identify at least two coexisting attractors: the chaotic trajectories [shown separately for Rössler and Lorenz subsystems in Figs. 5(g) and 5(h) respectively], and the fixed point attractor, in Figs. 5(e) and 5(f). Here the transition is directly from chaos to fixed point similar to Lorenz oscillator, Eq. (3) (neither is an UPO stabilized nor is a new periodic solution created) but in the present case the FP solution is created via riddling.

III. AMPLITUDE SUPPRESSION: ANALYTICAL ESTIMATION OF THE FREQUENCY JUMP

The phenomenon of frequency jump which can be observed (see Secs. II A and II B) in identical Rössler and Lorenz chaotic systems can be seen in coupled limit-cycles systems as well. Consider the case that has been studied in detail by Reddy *et al.* [5],

$$\begin{aligned} \dot{Z}_1(t) &= [1 + i\omega_1 - |Z_1(t)|^2]Z_1(t) + \epsilon[Z_2(t-\tau) - Z_1(t)], \\ \dot{Z}_2(t) &= [1 + i\omega_2 - |Z_2(t)|^2]Z_2(t) + \epsilon[Z_1(t-\tau) - Z_2(t)], \end{aligned} \quad (6)$$

where $Z_j(t)$, $j=1,2$ are the complex amplitudes of the j th oscillator with $|Z_j|=1$ and corresponding angular frequencies ω_j . ϵ is the coupling strength. For simplicity we consider identical oscillators with $\omega_1 = \omega_2$.

Specific results are shown for the case of $\omega_1=9$, which was the case used in Ref. [5]. The origin is a fixed point, i.e., $Z_j=0$. The spectrum of Lyapunov exponents and λ_{FP} , the exponent (the red line with circles) for the fixed point, for coupling strength $\epsilon=10$ is shown in Fig. 6(a). The resulting limit cycles at $\tau=0.05$ are in-phase, and at $\tau=0.25$ are out-of-phase, and these are shown in Figs. 6(b) and 6(c), respectively. The amplitude death region is in between $\tau \sim 0.1$ and 0.2 where all Lyapunov exponents are negative, $\lambda_1 = \lambda_2 = \lambda_{FP}$. The transient behavior of in- and out-of-phase motions in this amplitude death region are shown in Figs. 6(d) and 6(e) at $\tau=0.13$ and 0.19 , respectively. The transition from limit cycle to fixed point motion (say from $\tau=0.05$ to $\tau=0.14$) is happening at the Hopf bifurcation. The Lyapunov exponent of the fixed point, λ_{FP} , indicates this transition clearly, when it becomes negative.

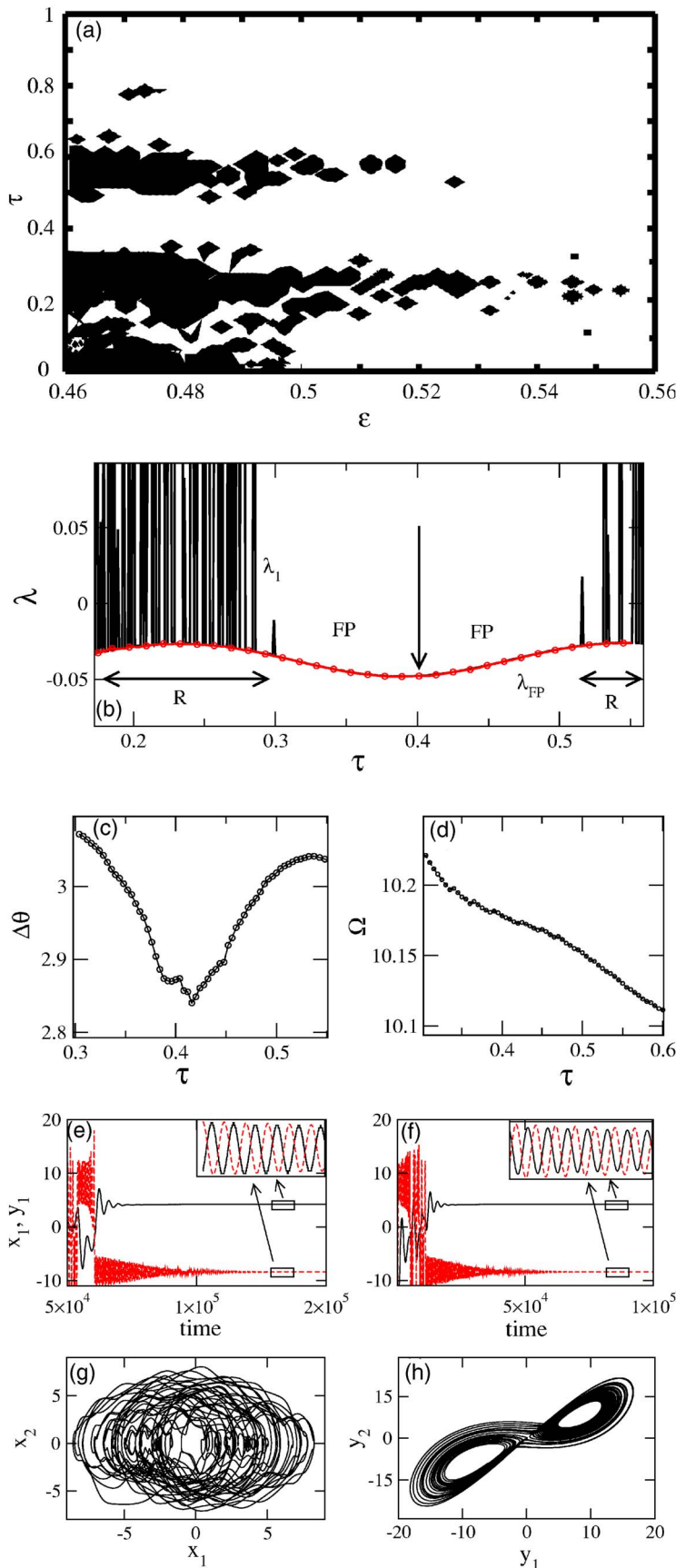


FIG. 5. (Color online) (a) Schematic phase diagram for the mixed Rössler and Lorenz system, Eq. (5), in the ϵ - τ plane. (b) The largest Lyapunov exponent (in black) as a function of time delay, τ , at fixed coupling strength $\epsilon=0.5$. R denotes regions of wild fluctuations in the Lyapunov exponent where chaotic and stable fixed point solutions coexist. The red line with circles represents the largest Lyapunov exponent of a fixed point ($x_{1*}=4.200308\dots$, $x_{2*}=-0.007246\dots$, $x_{3*}=-x_{2*}$, $y_{1*}=-8.406415\dots$, $y_{2*}=y_{1*}$, and $y_{3*}=26.500431\dots$). In contrast to previous examples, there is no discontinuous change in slope in the largest Lyapunov exponent. (c) The phase difference between oscillators with time delay τ . (d) Numerically calculated common frequency, Ω . Trajectories of Rössler (solid line) and Lorenz (dashed line) oscillators for (e) $\tau=0.35$ and (f) $\tau=0.45$. Chaotic trajectories of (g) Rössler and (h) Lorenz oscillators at $\tau=0.25$. The inset figures, which ordinate axes are scaled to see phase relations, are the enlarged view of corresponding marked boxes.

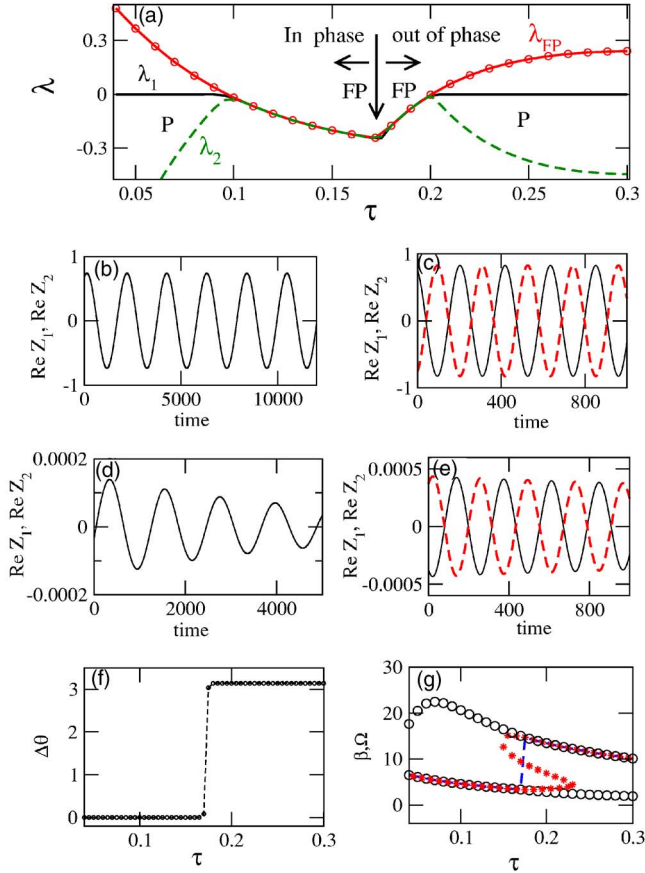


FIG. 6. (Color online) (a) Spectrum of Lyapunov exponents for the coupled limit-cycle oscillators (black and green correspond to λ_1 and λ_2 , respectively) with time delay, τ , at fixed coupling strength $\epsilon=10$. The red line with circles is the largest Lyapunov exponent for the fixed point, $Z_j=0$. In amplitude death region all Lyapunov exponents are negative and $\lambda_1=\lambda_2=\lambda_{FP}$. Trajectories of the oscillators 1 (solid line) and 2 (dashed line) with limit cycles at (b) $\tau=0.05$ (in-phase) and (c) $\tau=0.25$ (out-of-phase). $\text{Re } Z_j$ represents the real part for Z_j of the j th oscillator. Transient trajectories of oscillators 1 (solid line) and 2 (dashed line) with time at (d) $\tau=0.14$ (in-phase) and (e) $\tau=0.19$ (out-of-phase), respectively, for fixed point solutions. (f) The phase difference between oscillators with time delay τ which is equal to 0 and π before and after the transition, respectively. (g) Solution of Eqs. (8) (black circles) and (9) (red stars) with parameter τ . Blue dashed line indicates the numerically calculated common frequency, Ω .

Similar to Figs. 1(c) and 3(c), shown in Fig. 6(f) is the phase difference between oscillators of Eq. (6). The phases of the individual oscillators are defined here as $\theta_j = \tan^{-1}[\text{Im}(Z_j)/\text{Re}(Z_j)]$. The phase difference changes drastically from 0 to π when the largest Lyapunov exponent is minimum (the system is in most stable stat).

There is also a similar step transition in frequency across which in- and out-of-phase motions are present. The numerically calculated frequency, Ω , of Eq. (6) with time delay, is shown by the dashed line in Fig. 6(g). In order to understand this transition I consider the characteristic eigenvalue equation for Eq. (6) by assuming the linear perturbation to vary as $e^{\lambda t}$, which can be written as [5]

$$\lambda^2 - 2(a + i\omega)\lambda + (a^2 - \omega^2 + i2a\omega) - \epsilon^2 e^{-2\lambda\tau} = 0, \quad (7)$$

where $a=1-\epsilon$.

Letting $\lambda = \alpha + i\beta$ where α and β are real and imaginary part of the eigenvalues in Eq. (7), and separating real and imaginary parts leads to the equations

$$\alpha^2 - \beta^2 - 2(a\alpha - \beta\omega) + a^2 - \omega^2 - \epsilon^2 e^{-2\alpha\tau} \cos(2\beta\tau) = 0 \quad (8)$$

and

$$2\alpha\beta - 2(\alpha\omega + a\beta) + 2a\omega + \epsilon^2 e^{-2\alpha\tau} \sin(2\beta\tau) = 0. \quad (9)$$

As these equations are difficult to solve analytically we use the Lyapunov exponent of the fixed point, λ_{FP} , to find β numerically. The Lyapunov exponent of a fixed point is equal to the real part of the eigenvalue, i.e., $\lambda_{FP} = \alpha$, we use this in Eqs. (8) and (9). The resulting roots of these equations give β and are shown separately by circles and stars in Fig. 6(g). The final solution satisfying both these equations will be the solution of the eigenvalue equation, Eq. (7). Here we see that prior to the transition, the smaller root of Eq. (8) matches Eq. (9) while after the transition it is the larger root. The agreement with β computed directly from the eigenvalue equation and numerically calculated frequency (dashed line) is excellent.

This analysis has been possible for the case of coupled limit cycle oscillators. Although it cannot be extended to the case of chaotic systems, it is possible to conjecture that the underlying mechanism could be similar, namely that the transition is due to a jump in the imaginary part of the eigenvalue of the fixed point.

IV. DISCUSSION AND SUMMARY

In this paper I have shown that the phenomenology developed for the cause of amplitude death in systems with limit-cycle oscillators [1–3] can be extended to the case of chaotic dynamical systems. This also extends earlier work on amplitude modulation [29], where time-delay coupling has been used to stabilize undesirable low-frequency chaotic fluctuations in a semiconductor laser. (There it was seen that time-delay coupling could convert chaotic oscillations of the laser field with power dropouts into quasiperiodic motion without power dropouts.)

Amplitude death in chaotic systems is a consequence of time delays that stabilize fixed points that might either have existed in the uncoupled system, or that might have been created through the interaction. The evidence presented here is largely numerical. We considered separately the cases of the interacting systems being (a) identical, (b) the same but with mismatched parameters, and (c) completely distinct. In all cases, amplitude death can occur, and moreover, this can occur over a range of parameter values. The mechanics through which chaos converts into fixed point motion are different in these examples: in some cases it happens via limit cycle at Hopf bifurcation while other cases fixed point gets stabilized directly. In mixed systems this stabilization is initiated via riddling phenomenon.

We have used the Lyapunov exponent as an indicator to detect the different dynamical regimes, and to detect transi-

tions, as for example, in the case of identical oscillators, when the subsystems are oscillating in- or out-of-phase with each other. When the subsystems are maximally stable, namely the largest Lyapunov exponent is minimum, a “phase” transition occurs, and motion goes from being in-phase to being out-of-phase. Measures that detect this transition are the phase difference and frequency; although not possible to analyze in chaotic systems, it is possible to get some understanding of this transition for a limit-cycle case. Apart from its mathematical interest, the possibility of using a transition from dynamics on a limit cycle which is in-phase to that which is out-of-phase in practical applications suggests itself. In experimental application of amplitude death phenomena, where transient dynamics always persists, it will be important to know the parameters where motion is either in-phase (low frequency) or out-of-phase (high frequency). An important possible use of it could be coupled chaotic

lasers [29], where a relatively higher degree of constant output could be obtained by keeping the parameters of the lasers and the time-delay coupling in the out-of-phase regime compared to that of the in-phase [25].

Experimental verification of this generalization of the conventional amplitude-death phenomenon that occurs in limit-cycle oscillators may in fact prove to be technically simple to achieve since it persists even after parameters mismatch, and occurs for different types of coupled oscillators, bypassing the need for irregular oscillations (e.g., Ref. [29]).

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